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Isospectral problem: interplay between Liouville equations, Darboux transforms and McKean–Trubowitz flows

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Abstract. Systems of Liouville equations are generated by isospectral transforms of the Schrödinger operator and describe the isospectral evolution of both a potential and eigenfunctions. Relations of these equations with the McKean–Trubowitz isospectral flows are studied. The role of these flows among other isospectral flows is discussed.

1. Introduction

The Darboux transform [1] is well known in mathematical physics. If a solution $\psi^0(x)$ of the Schrödinger equation for a given operator $\hat{\boldsymbol{H}}^0$ is known, then this transform provides a family of operators $\hat{\boldsymbol{H}}$ whose spectrum coincides with that of $\hat{\boldsymbol{H}}^0$, except for one (added or removed) eigenvalue. If all the eigenfunctions of $\hat{\boldsymbol{H}}^0$ are known, then the Darboux transform allows us to get eigenfunctions of the operator $\hat{\boldsymbol{H}}$ [2].

Thus, the Darboux transform is closely related to the notion of *an isospectral transform* of the Schrödinger operator.

The relations of this transform and its generalizations [3, 4] with other approaches that allow one to construct families of isospectral Schrödinger operators were analysed in [5] (see also [6] and the bibliography therein). The factorization of the Schrödinger operator [7, 8], also related to the Darboux transform, is successfully used in the isospectral problem [9]; in its turn, it provides a basis for the supersymmetry theory [10, 11] and the dressing chain method [12]. Naturally, I would like to note also a method that uses nonlinear Fock shift operators [13].

Traditionally, the Darboux transform is not associated with any *evolutionary* formulation of the isospectral problem (the deformation of the potential as well as the eigenfunctions does not obey any partial differential equations).

On the other hand, isospectral properties of the Schrödinger operator play an extremely important role in the soliton theory (in particular, in the inverse scattering transform) [14, 15], where they are related to the existence of soliton solutions of integrable evolution equations; the list of such equations is well known [16].

It has been known since Poincaré [17] that the stationary Schrödinger equation is equivalent to the integrable Liouville equation $S_{xt} = \exp S$ where t is an implicit parameter of the Schrödinger equation, which 'specifies' its particular solutions for one and the same potential. Recently the relation between these two equations was analysed in detail and generalized by Santini in [18]. In this paper we discuss how this simple relation appears

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in isospectral transforms of the Schrödinger equation and allows one to obtain Liouville equations for more complex isospectral 'evolution' of eigenfunctions and of a potential.

Recently, it was discovered [19] that there exists another branch of the isospectral problem, which also leads to Liouville equations for a deformation of eigenfunctions and a potential of the Schrödinger operator. It is related to an analysis of the flows that were studied in detail by McKean and Trubowitz [20]:

$$U_t(x,t) = \frac{\partial}{\partial x}\frac{\delta \hat{\boldsymbol{H}}}{\delta U} = \sum_k \tau_k[\psi_k^2(x,t)]_x. \tag{1.1}$$

(ψ_k are eigenfunctions of the operator $\hat{\boldsymbol{H}} = -\frac{1}{2}\nabla_x^2 + U(x,t)$; τ_k are numbers, t is a parameter.) As was shown in [20], such flows are associated with an isospectral deformation of the eigenfunctions and of the potential in the parameter t (many mathematical aspects of this deformation were later thoroughly studied by Levitan [21]). The authors of [19] have found that this deformation is governed by a system of coupled evolutional equations of the Liouville kind; in particular, in the case of 'individual flow' (in terms of [20]; $\tau_n = 1$; $\tau_k = 0$, $k \neq n$) the integrable Liouville equation arises. The 'two-level' flow ($\tau_n \neq 0$ and $\tau_m \neq 0$; $\tau_k = 0$, $k \neq n, m$) is studied in detail in [22]; it is shown there that in this case the coupled evolutional equations can be split into two independent integrable Liouville equations for special combinations of the eigenfunctions ψ_n, ψ_m and their derivatives; the same equation governs a certain function of the potential $U(x,t)$.

In this paper we try to establish relations among the above-listed approaches to the isospectral problem for the Schrödinger operator: the Darboux transform, the 'evolutional approach' leading to the Liouville equations, and the McKean–Trubowitz flows (in what follows they will be referred to as 'MKT flows').

The paper is arranged as follows. In section 2 we briefly recall a correspondence between the Schrödinger equation and the Liouville equation; we also establish relations of functions that are obtained by the Darboux transform of solutions to the Schrödinger equation, with solutions of the Liouville equation. In sections 3 and 4 these relations are illustrated by examples of simple and double ('cross') Darboux transforms. In both the cases the Liouville equations that govern the isospectral deformation are derived; they coincide with the equations that arise in the 'evolutional approach' if the MKT flows are used. In the concluding section we discuss the place of the MKT flows (and, therefore, of the Liouville equations) in the general context of the isospectral problem.

2. Relation of Liouville equations to the Darboux transform for Schrödinger operators

Let us recall a relation between the Schrödinger and Liouville equations [17, 18]. Write down the Schrödinger equation for an arbitrary triplet $\{\Phi(x,t), E(t), V(x,t)\}$ (t is a parameter):

$$-\frac{1}{2}\frac{\Phi_{xx}}{\Phi} + V = E. \tag{2.1}$$

Demand that parameter E and potential V do not depend on t ($E =$ constant; $V(x,t) \equiv V(x)$); then

$$\left(\frac{\Phi_{xx}}{\Phi}\right)_t = 0. \tag{2.2}$$

This equation can be equivalently rewritten as an equation for the function $F \equiv 1/\Phi^2$:

$$\left(\frac{F_x}{F}\right)_t = h(t)F \tag{2.3}$$

where $h(t)$ is an arbitrary function (of course, t can be renormalized, so $h(t)$ can be set to ± 1). Equation (2.3) reduces to the classical form of the integrable Liouville equation $S_{xt} = \exp S$ by a simple substitution of the type $F \longrightarrow \exp S$; for brevity, we will also call it 'the Liouville equation'.

For convenience, let us introduce an operator of 'logarithmic derivative' $\mathrm{D}_{\log}$: $\mathrm{D}_{\log} b \equiv b_x/b \equiv (\ln|b|)_x$; so the Liouville equation (2.3) can be written in the form

$$\frac{\partial}{\partial t}\mathrm{D}_{\log} F = h(t)F.$$

Naturally, solutions of equations (2.2) and (2.3) are also in correspondence to each other. Solving (2.2), we get a well known general two-parameter family of solutions to the Schrödinger equation with *fixed* potential V and parameter E:

$$\Phi^{(\alpha,\gamma)}(x,t) = \Phi^0(x)\gamma(t)\left(1 + \alpha(t)\int_x^\infty \frac{\mathrm{d}x'}{[\Phi^0]^2}\right) \tag{2.4}$$

where $\Phi^0(x)$ is any particular solution to this equation (with $\alpha = 0$). It is easy to verify that the substitution $1/\Phi^2 \Rightarrow F$ in this expression gives us a general solution to the Liouville equation (2.3):

$$F^{(\alpha,\gamma)}(x,t) = \frac{F^0(x)}{\gamma^2(t)} \frac{1}{[1 + \alpha(t)\int_x^\infty F^0\,\mathrm{d}x']^2} \tag{2.5}$$

where one should identify $\gamma^2(t)$ with $h(t)/2\alpha'(t)$. It is clear that in the family of solutions (2.4) only the parameter $\alpha(t)$ is essential: since we deal with a linear ODE, the second parameter determines a magnitude of a function. Setting $\gamma = 1$ and choosing $h(t) = 2\alpha'(t)$, we get a correspondence between *one-parameter* families $\Phi^{(\alpha)}$ and $F^{(\alpha)}$ of solutions to the Schrödinger and Liouville equations.

Thus, the evolution of the function $F^{(\alpha)} = 1/[\Phi^{(\alpha)}]^2$ in t obeying the Liouville equation (2.3) corresponds to a continuous variation of the parameter $\alpha(t)$ in the family $\Phi^{(\alpha)}$ of solutions to the Schrödinger equation with *fixed* potential (this potential is determined by the function $\Phi^{(0)}$).

Now consider how the above correspondence manifests itself in isospectral transforms of the Schrödinger operator. Let Φ_n be solutions (not eigenfunctions) of the Schrödinger equation for a certain fixed potential $V(x)$ that correspond to a set of parameters $\{E_n\}$. Consider any transform $\boldsymbol{T}$ that translates solutions Φ_n to *eigenfunctions* ψ_n of the Schrödinger operator with another potential U and respective *eigenvalues* E_n: $\boldsymbol{T}[\Phi_n] \Rightarrow \psi_n$; relation between potentials U and V can also be written as a transform: $\tilde{\boldsymbol{T}}[V] = U$. Let the transform $\boldsymbol{T}$ be N-parameter with parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$. Demand that for the Schrödinger operator with new potential U the rest of the spectrum does not depend on these parameters. Thus we have an *implicit* N-parameter isospectral deformation of potential $U = U(x;\alpha_1,\ldots,\alpha_N)$ and, accordingly, an isospectral deformation of eigenfunctions $\psi_n = \psi_n(x;\alpha_1,\ldots,\alpha_N)$ (figure 1).

If an inverse transform $\boldsymbol{T}^{-1}$ is defined then the isospectral deformation determined by $\boldsymbol{T}$ can be made *explicit*:

$$\begin{aligned}\psi_n(x;\alpha_1,\ldots,\alpha_N) &= \boldsymbol{T}_{(\alpha_1,\ldots,\alpha_N)}[\Phi_n(x)] \\ &= \boldsymbol{T}_{(\alpha_1,\ldots,\alpha_N)}\boldsymbol{T}^{-1}_{(\beta_1,\ldots,\beta_N)}[\psi_n(x;\beta_1,\ldots,\beta_N)].\end{aligned} \tag{2.6}$$

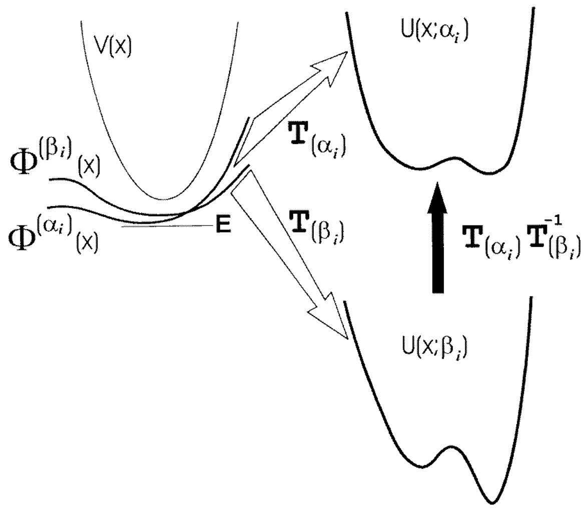

Figure 1. Isospectral transform (vertical fat arrow) generated by a family of solutions to the Schrödinger equation with a fixed potential and parameter E (schematically).

Let transform $\boldsymbol{T}$ be determined by N solutions Φ_n to the Schrödinger equation for potential V, which correspond to an arbitrary set $\{E_n; n = k_1, k_2, \ldots, k_N\}$; furthermore, let the parametrization of this transform be determined by the parameters $\{\alpha_n; n = k_1, k_2, \ldots, k_N\}$ of those essentially one-parameter solutions. We know already that all the functions $F_n = 1/\Phi_n^2 = 1/(\boldsymbol{T}^{-1}[\psi_n])^2$ 'evolve' obeying N Liouville equations

$$\frac{\partial}{\partial t_n}\mathrm{D}_{\log} F_n = \varepsilon_n F_n \qquad n = 0, 1, \ldots \qquad \varepsilon_n = \pm 1 \tag{2.7}$$

each with its own 'time' $t_n = t_n(\alpha_n)$. Note, that one can take a common time in these equations, introducing parameters τ_n: $t_n = 2\tau_n t$:

$$\frac{\partial}{\partial t}\mathrm{D}_{\log} F_n = 2\varepsilon_n \tau_n F_n \qquad n = 0, 1, \ldots \qquad \varepsilon_n = \pm 1. \tag{2.8}$$

(The number coefficient is introduced to compare below with already known results.)

We can summarize the above reasonings in the form of the following

Statement. If the transform $\boldsymbol{T}$ is defined as stated above, then the evolution of the eigenfunctions $\psi_n(x; \{\alpha_i\}) = \boldsymbol{T}_{\{\alpha_i\}}\boldsymbol{T}^{-1}_{\{\beta_i\}}[\psi_n(x; \{\beta_i\})]$ of the Schrödinger operator with potential $U(x; \{\alpha_i\}) = \tilde{\boldsymbol{T}} V(x)$ in their parameters is isospectral, and the dependence of each function $F_n(x; \alpha_n) = 1/(\boldsymbol{T}^{-1}[\psi(x; \{\alpha_i\})])^2$ on its *only* parameter α_n obeys the Liouville equation.

3. Simple Darboux transform and 'individual' MKT flow

Consider a stationary potential V: $V_t = 0$. Fix a value of E_n and take some solution Φ_n^0 of the Schrödinger equation with the potential V for the parameter $E = E_n$ (later on this value will become an nth eigenvalue). Let Φ_n^0 satisfy the following boundary conditions:

$\Phi_n^0(x \to \pm\infty) \to \pm\infty$; no other restrictions are now put on Φ_n^0. The associated one-parameter solution is

$$\Phi_n^{(\alpha)} = \Phi_n^0\left(1 + \alpha \int_x^\infty \frac{dx'}{[\Phi_n^0]^2}\right). \tag{3.1}$$

As an isospectral transform we will use the simple Darboux transform, determined by the function $\Phi_n^{(\alpha)}$; this function itself is transformed in this procedure as follows:

$$\Phi_n^{(\alpha)} \Rightarrow \psi_n^{(\alpha)} = \frac{1}{\Phi_n^{(\alpha)}} = \frac{1}{\Phi_n^0(1 + \alpha \int_x^\infty dx' \, [\Phi_n^0]^{-2})} = \frac{\psi_n^0}{1 + \alpha \int_x^\infty [\psi_n^0]^2 \, dx'}. \tag{3.2}$$

The new function $\psi_n^{(\alpha)}$ is an *eigenfunction* of the Schrödinger operator with the new one-parameter potential

$$U^{(\alpha)} = V - \frac{\partial}{\partial x} \mathrm{D}_{\log} \Phi_n^{(\alpha)} \tag{3.3}$$

for the same value E_n (which is now an eigenvalue).

It should be noted here that we use the Darboux transform formally; we do not dwell on singularities in Φ_n as well as in V (which are inevitable if we take eigenvalues E_n with $n > 0$, so that ψ_n has zeros). But the final evolutional equations that are obtained as a result of this procedure coincide with those derived in [19, 22] using another approach (this will be discussed below) where they are proved to be correct for arbitrary n; which allows one to believe that such a free usage of the Darboux transform is not dangerous here. Note also that a profound study of singularities that appear in the Backlund transforms can be found in the paper [23].

Having made these remarks, return to formulae (3.2), (3.3). Now let us use the following fact: since $\Phi_n^{(\alpha)}$ is a one-parameter solution of the Schrödinger equation with the fixed potential V, the function $F_n = 1/\{\Phi_n^{(\alpha)}\}^2$ is a solution to the Liouville equation $(\mathrm{D}_{\log} F_n)_t = 2\tau_n F_n$, where $t \equiv \alpha$.

This means that the function $\{\psi_n^{(\alpha)}\}^2 = 1/\{\Phi_n^{(\alpha)}\}^2$ is exactly a solution to the Liouville equation:

$$\frac{\partial}{\partial t} \mathrm{D}_{\log} \{\psi_n^{(\alpha)}\}^2 = 2\tau_n \{\psi_n^{(\alpha)}\}^2. \tag{3.4}$$

Thus, we have found that the isospectral deformation specified by the Darboux transform can be treated as being induced by the Liouville equation (3.4) for the function ψ_n^2.

Now let us construct an expression for U_t. The potential U depends on t through the parameter $\alpha(t) \equiv t$ of the Darboux transform; see equation (3.3). Then for U_t we get the expression:

$$U_t = -\frac{\partial^2}{\partial x \partial t} \mathrm{D}_{\log} \Phi_n^{(\alpha)}. \tag{3.5}$$

On the other hand, we have the Liouville equation $(\mathrm{D}_{\log} F_n)_t = 2\tau_n F_n$ for the function $F_n = 1/\{\Phi_n^{(\alpha)}\}^2$, which can be rewritten in the form

$$\frac{\partial}{\partial t} \mathrm{D}_{\log} \Phi_n^{(\alpha)} = -\frac{\tau_n}{[\Phi_n^{(\alpha)}]^2} \tag{3.6}$$

so we obtain

$$U_t = \frac{\partial}{\partial x} \left\{ \frac{\tau_n}{[\Phi_n^{(\alpha)}]^2} \right\}. \tag{3.7}$$

Since $\Phi_n^{(\alpha)} = 1/\psi_n^{(\alpha)}$, we can define U_t in a self-consistent way through the eigenfunction ψ_n of the Schrödinger operator with potential U:

$$U_t = \tau_n\{\psi_n^2\}_x. \tag{3.8}$$

This formula is well known: it determines the 'individual' MKT flow (1.1) [20]. And otherwise, as shown in [19], it is equation (3.4) that describes the isospectral evolution of the eigenfunction ψ_n in the case of the 'individual' MKT flow. Thus, in the case $N = 1$ we have established a direct correspondence of the above scheme (which uses the relation between the Darboux transform of the general solution to the Schrödinger equation and the Liouville equation) to the 'evolutional' description [19] of the isospectral deformations that are defined by the MKT flows.

4. The cross Darboux transform and two-level MKT flow

Consider a *pair* of solutions $\Phi_{n,m}^0$ to the Schrödinger equation for a stationary potential V. Associated one-parameter solutions are given by the expressions:

$$\Phi_n^{(\alpha)} = \Phi_n^0\left(1 + \alpha\int_x^\infty \frac{\mathrm{d}x'}{[\Phi_n^0]^2}\right) \tag{4.1}$$

$$\Phi_m^{(\beta)} = \Phi_m^0\left(1 + \beta\int_x^\infty \frac{\mathrm{d}x'}{[\Phi_m^0]^2}\right). \tag{4.2}$$

Let us make a *cross* Darboux transform ('the **X**-Darboux transform') on the pair $\Phi_n^{(\alpha)}$, $\Phi_m^{(\beta)}$; the formulae of this transform have the form:

$$\Phi_n^{(\alpha)} \Rightarrow \psi_n^{(\alpha,\beta)} = -\frac{\Phi_m^{(\beta)}}{[\Phi_n^{(\alpha)}, \Phi_m^{(\beta)}]} \tag{4.3}$$

$$\Phi_m^{(\beta)} \Rightarrow \psi_m^{(\alpha,\beta)} = +\frac{\Phi_n^{(\alpha)}}{[\Phi_n^{(\alpha)}, \Phi_m^{(\beta)}]}. \tag{4.4}$$

(Here the standard notation for the Wronskian of two functions was used: $[a, b] \equiv ab_x - a_x b$.) Substituting expressions (4.1), (4.2) for the one-parameter solutions in the right-hand sides of equations (4.3), (4.4), we get the relations:

$$\psi_n^{(\alpha,\beta)} = f_n(\Phi_n^0, \Phi_m^0; \alpha, \beta) \tag{4.5}$$

$$\psi_m^{(\alpha,\beta)} = f_m(\Phi_n^0, \Phi_m^0; \alpha, \beta). \tag{4.6}$$

In turn, the functions Φ_n^0, Φ_m^0 are related to the functions ψ_n^0, ψ_m^0 by the inverse **X**-Darboux transform:

$$\psi_n \Rightarrow \Phi_n = +\frac{\psi_m}{[\psi_n, \psi_m]} \tag{4.7}$$

$$\psi_m \Rightarrow \Phi_m = -\frac{\psi_n}{[\psi_n, \psi_m]} \tag{4.8}$$

(hereafter for brevity the superscripts (α, β) in right-hand sides are omitted), so relations (4.5), (4.6) can be written in the form:

$$\psi_n^{(\alpha,\beta)} = \tilde{f}_n(\psi_n^0, \psi_m^0; \alpha, \beta) \tag{4.9}$$

$$\psi_m^{(\alpha,\beta)} = \tilde{f}_m(\psi_n^0, \psi_m^0; \alpha, \beta). \tag{4.10}$$

Since the dependencies $\tilde{f}_n$, $\tilde{f}_m$ are quite cumbersome, we will not write down explicit expressions.

The functions $\psi_n^{(\alpha,\beta)}$, $\psi_m^{(\alpha,\beta)}$ will be eigenfunctions of the Schrödinger operator with the *two*-parameter potential

$$U^{(\alpha,\beta)} = V - \frac{\partial}{\partial x}\mathrm{D}_{\log}[\Phi_n^{(\alpha)}, \Phi_m^{(\beta)}]. \tag{4.11}$$

On the other hand, since the functions $\Phi_n^{(\alpha)}$, $\Phi_m^{(\beta)}$ are one-parameter solutions to the Schrödinger equation with the fixed potential V, the functions $F_n \equiv [\Phi_n^{(\alpha)}]^{-2}$ and $F_m \equiv [\Phi_m^{(\beta)}]^{-2}$ are solutions to the corresponding Liouville equations:

$$\frac{\partial}{\partial t_n}\mathrm{D}_{\log}F_n = \varepsilon_n F_n \qquad \varepsilon_n = \pm 1 \tag{4.12}$$

$$\frac{\partial}{\partial t_m}\mathrm{D}_{\log}F_m = \varepsilon_m F_m \qquad \varepsilon_m = \pm 1 \tag{4.13}$$

$t_n = 2\alpha$, $t_m = 2\beta$. Using the formulae for the inverse **X**-Darboux transform (4.7), (4.8) we can express the functions $F_{n,m}$ in terms of the transformed functions $\psi_n^{(\alpha,\beta)}$, $\psi_m^{(\alpha,\beta)}$; then the Liouville equations for $F_{n,m}$ can be rewritten as follows:

$$\frac{\partial}{\partial t}\mathrm{D}_{\log}\left\{\frac{[\psi_n, \psi_m]}{\psi_n}\right\}^2 = c_n\varepsilon_n\left\{\frac{[\psi_n, \psi_m]}{\psi_n}\right\}^2 \tag{4.14}$$

$$\frac{\partial}{\partial t}\mathrm{D}_{\log}\left\{\frac{[\psi_n, \psi_m]}{\psi_m}\right\}^2 = c_m\varepsilon_m\left\{\frac{[\psi_n, \psi_m]}{\psi_m}\right\}^2 \tag{4.15}$$

where the common 'time' t is introduced by the scaling transforms $t = t_n/c_n = t_m/c_m$; $c_n, c_m =$ constant.

Thus, in the case of the cross Darboux transform the isospectral deformation of the eigenfunctions corresponds to the evolution of mixed functions $([\psi_n, \psi_m]/\psi_m)^2$ and $([\psi_n, \psi_m]/\psi_n)^2$ that obeys two independent Liouville equations that can be easily integrated.

Now let us build an expression for U_t in this case and see which flow is related with the above scheme 'two one-parameter solutions of the Schrödinger equation for a generating potential + the **X**-Darboux transform $\Longrightarrow$ two integrable Liouville equations'.

The **X**-Darboux transform of the potential V is executed in accordance with equation (4.11); differentiating it by t and rearranging (taking into account the Schrödinger equation) we find that the expression for U_t has the form

$$U_t = -2\Delta_{nm}\frac{\partial}{\partial x}\left\{\left(\frac{\Phi_n\Phi_m}{[\Phi_n, \Phi_m]}\right)^2\frac{\partial}{\partial t}[\mathrm{D}_{\log}\Phi_n - \mathrm{D}_{\log}\Phi_m]\right\} \tag{4.16}$$

(for brevity superscripts are omitted; $\Delta_{nm} \equiv E_n - E_m$). The equations for the functions Φ_n and Φ_m can be written in the form

$$\frac{\partial}{\partial t}\mathrm{D}_{\log}\Phi_n = -\frac{\tau_n}{2\Delta_{nm}}\frac{1}{\Phi_n^2} \tag{4.17}$$

$$\frac{\partial}{\partial t}\mathrm{D}_{\log}\Phi_m = +\frac{\tau_m}{2\Delta_{nm}}\frac{1}{\Phi_m^2}. \tag{4.18}$$

Here we used an arbitrariness of the functions $h_{n,m}(t)$ in equations (2.3) (and, thus, our freedom in choosing the constants and the signs in the right-hand sides when we scale the time); the signs are to meet the condition that those equations transform to each other by the change $n \Leftrightarrow m$ (taking into account that $\Delta_{nm} = -\Delta_{mn}$). Then we get

$$U_t = \frac{\partial}{\partial x}\left\{\left(\frac{\Phi_n\Phi_m}{[\Phi_n, \Phi_m]}\right)^2\left[\frac{\tau_n}{\Phi_n^2} + \frac{\tau_m}{\Phi_m^2}\right]\right\}$$

$$= \frac{\partial}{\partial x}\left\{\tau_n\left(\frac{\Phi_m}{[\Phi_n,\Phi_m]}\right)^2 + \tau_m\left(\frac{\Phi_n}{[\Phi_n,\Phi_m]}\right)^2\right\}. \tag{4.19}$$

Finally, using the formulae for the double Darboux transform from $\Phi_{n,m}$ to $\psi_{n,m}$, we find a self-consistent expression for U_t:

$$U_t = \tau_n\{\psi_n^2\}_x + \tau_m\{\psi_m^2\}_x. \tag{4.20}$$

Thus, in the case $N = 2$ as well as in the case $N = 1$ the self-consistent expression for U_t that arises in the scheme using the Liouville equations is exactly the expression for the McKean–Trubowitz flow.

On the other hand, it can be shown [22], that equations (4.14), (4.15) (with $\varepsilon_n\varepsilon_m < 0$) arise as a result of splitting the coupled evolutional equations that describe the isospectral deformation of the functions $\psi_n(x,t)$, $\psi_m(x,t)$ [19], if that deformation is defined by the MKT flow (4.20).

Solutions of these equations written in the form (4.17), (4.18) are as follows:

$$\frac{1}{\Phi_n^2} = +2\frac{\partial}{\partial t_1}\mathrm{D}_{\log}\left[1 - \frac{1}{2}\int_0^{t_1}\exp g_1(t')\,\mathrm{d}t'\int_x^\infty \frac{1}{\Phi_n^2}\bigg|_{t=0}\mathrm{d}x'\right] \tag{4.21}$$

$$\frac{1}{\Phi_m^2} = -2\frac{\partial}{\partial t_2}\mathrm{D}_{\log}\left[1 + \frac{1}{2}\int_0^{t_2}\exp g_2(t')\,\mathrm{d}t'\int_x^\infty \frac{1}{\Phi_m^2}\bigg|_{t=0}\mathrm{d}x'\right] \tag{4.22}$$

where $t_1 = \tau_n/\Delta_{nm}\ t$, $t_2 = \tau_m/\Delta_{nm}\ t$; g_1 and g_2 are arbitrary functions of t. Denoting the right-hand sides of these solutions by F_1 and F_2, respectively, one can write final formulae for the isospectral deformation of the eigenfunctions in the case of the two-level MKT flow:

$$\psi_n = \frac{1}{\sqrt{F_1}\int_x^\infty \frac{\mathrm{d}x'}{\sqrt{F_1F_2}}} \qquad \psi_m = \frac{1}{\sqrt{F_2}\int_x^\infty \frac{\mathrm{d}x'}{\sqrt{F_1F_2}}}. \tag{4.23}$$

Thus, the direct correspondence between the above scheme 'one-parameter solutions of the Schrödinger equation + the Darboux transform ⇒ the Liouville equations' and the evolutional scheme, in which the isospectral evolution is specified by the MKT flows, is established in the cases $N = 1, 2$. It can be assumed with great confidence that this correspondence is valid in the general case as well.

5. Conclusion: McKean–Trubowitz flows among other isospectral flows

Let us try to clearly ascertain the place of the McKean–Trubowitz flows in the general isospectral problem. Consider an arbitrary relation of the form:

$$U_t = F(U, U_x, \ldots; \{\psi_n\}, \{(\psi_n)_x\}, \ldots) \tag{5.1}$$

where ψ_n are eigenfunctions of the Schrödinger operator with potential U. Assume that the function F can be represented as a flow density, i.e. as a gradient of a variational derivative for a certain functional. Then one can treat relation (5.1) as a flow. Under which conditions is this flow related to an *isospectral* deformation of a potential and eigenfunctions in the parameter t? As can be easily derived from (2.1), the condition

$$(E_n)_t = 0 \qquad \forall t \tag{5.2}$$

leads to the system of equations

$$\int_{-\infty}^{+\infty} U_t\psi_n^2 = 0 \qquad n \geqslant 0 \tag{5.3}$$

so, all functionals F that specify isospectral flows are to satisfy the following system of equations:

$$\int_{-\infty}^{+\infty} F\psi_n^2 = 0 \qquad n \geqslant 0. \tag{5.4}$$

In other words, all isospectral flows are to belong to the orthogonal complement to the set $\{\psi_n^2\}$.

Differentiating (2.1), we get the following equations:

$$\tfrac{1}{8}(\psi_n)^2_{xxx} - \tfrac{1}{2}U_x\psi_n^2 - (U - E_n)(\psi_n^2)_x = 0 \qquad n \geqslant 0 \tag{5.5}$$

which can be rewritten in the form

$$\boldsymbol{L}[\psi_n^2]_x = E_n[\psi_n^2]_x \qquad n \geqslant 0 \tag{5.6}$$

if one introduces the integro-differential operator

$$\boldsymbol{L} \equiv -\tfrac{1}{8}D^2 + \tfrac{1}{2}U_x D^{-1} + U \tag{5.7}$$

where $Dg(x) \equiv \mathrm{d}g/\mathrm{d}x$, $D^{-1}g(x) \equiv \int_{-\infty}^{x} g(x')\,\mathrm{d}x'$ (this observation is usually assigned to Hermit [24]).

Operator $\boldsymbol{L}$ has two important properties†.

(1) The functions $[\psi_n^2]_x$ are its eigenfunctions; they correspond to the eigenvalues E_n; this fact is represented by formula (5.6). Thus, if the set of pairs $\{(E_m, \psi_m)\}$ is a set of eigenelements for the operator $H = -\frac{1}{2}D^2 + U$, then the set $\{(E_m, [\psi_m^2]_x)\}$ is a set of eigenelements for the operator $\boldsymbol{L}$. Note that $\boldsymbol{L}$ is not an Hermitian operator, so the set $[\psi_n^2]_x$ does not have to be complete.

(2)

$$\int_{-\infty}^{+\infty} \psi_n^2 \boldsymbol{L} f(x)\,\mathrm{d}x = E_n \int_{-\infty}^{+\infty} \psi_n^2 f(x)\,\mathrm{d}x \qquad n \geqslant 0 \tag{5.8}$$

for a wide class of functions $f(x)$ (this class includes functions that grow at infinity slower than the exponential). This implies that if some relation $U_t = F$ defines an isospectral flow, then all the flows $U_t = \boldsymbol{L}^k F$, $k > 0$ are also isospectral, since conditions (5.3) are satisfied for all of them; this fact is used in constructing hierarchies of integrable evolutional equations. Because of this property the operator $\boldsymbol{L}$ is often called *the recursion operator*‡.

Compare two elementary isospectral flows (this implies that they satisfy equation (5.3)). The first of them is the shift flow

$$U_t = cU_x. \tag{5.10}$$

It can be shown that $\int_{-\infty}^{+\infty} U_x\psi_n^2\,\mathrm{d}x$ always vanishes (in the same class of potentials that grow slower than the exponential); this simple result immediately follows from equation (2.1). But usually, to avoid a singularity in t, one requires a 'good' behaviour of U_x at infinity (in particular, the condition $U_x \to 0$ as $x \to \pm\infty$ specifies the class of scattering potentials).

Starting with the shift flow (5.10) and sequentially applying the recursion operator $\boldsymbol{L}$, one builds a hierarchy of KdV flows: $U_t = c_0U_x$, $U_t = c_1(U_{xxx} + 6(U^2)_x), \ldots, U_t = c_m\boldsymbol{L}^m U_x$. Evidently, all those flows lie in the orthogonal complement to the set $\{\psi_n^2\}$.

† A wide list of its properties can be found in [16].

‡ Note that the operator $\boldsymbol{L}$ can be written in the following simple symmetric form:

$$\boldsymbol{L} = -\tfrac{1}{8}D^2 + \sqrt{U}D\sqrt{U}D^{-1}. \tag{5.9}$$

(Here U is assumed to be positive; the generalization to an arbitrary potential is evident.)

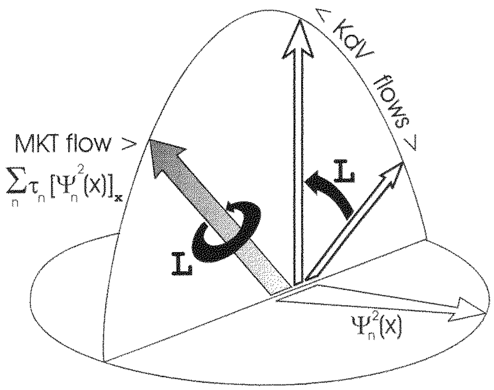

Figure 2. Isospectral flows: the McKean–Trubowitz flow as an 'eigenflow' of the recursion operator $\boldsymbol{L}$.

The second 'elementary' type of isospectral flows is that studied by McKean and Trubowitz in [20]:

$$U_t = \sum_n \tau_n [\psi_n^2]_x \tag{5.11}$$

(τ_n are arbitrary numbers); it is not difficult to prove that

$$\int_{-\infty}^{+\infty} \psi_n^2 [\psi_m^2]_x \, \mathrm{d}x = 0 \qquad n, m \geqslant 0 \tag{5.12}$$

by virtue of equation (2.1). In other words, the sets $\{\psi_n^2\}$ and $\{[\psi_n^2]_x\}$ lie in orthogonal subspaces; thus the necessary condition of isospectrality for the flows (5.11) is satisfied.

Note two important properties of MKT flows.

(a) They are defined on eigenfunctions (which always 'behave well'), without using the potential. Hence, such flows exist for potentials that grow at infinity (e.g. for the harmonic oscillator) as well as for scattering potentials. The isospectral problem for both types of potentials was analysed in [19]; it was shown there, that the deformation of a potential induced by those flows involves a splitting off of a local potential well that asymptotically takes a reflectionless soliton form ($-\gamma^2 \cosh^{-2} \gamma\xi$). It is of interest, that this result holds for a wide class of models, including the harmonic oscillator; a particular choice of a potential determines only how the parameters of this soliton depend on the parameter t.

(b) As follows from the first property of the recursion operator $\boldsymbol{L}$, the functions $[\psi_n^2]_x$ are its eigenfunctions. This means that, starting with this flow, one cannot build an $\boldsymbol{L}$-hierarchy: the operator $\boldsymbol{L}$ simply changes the'weights' τ_n, but the flow remains related to the linear envelope $\{[\psi_n^2]_x\}$. In contrast, elements of a function space that are associated with different flows of the KdV hierarchy, are sequentially mapped onto each other by the operator $\boldsymbol{L}$ (figure 2).

Thus, 'from the viewpoint' of the recursion operator $\boldsymbol{L}$, the MKT flows can be called its 'eigenflows'; this explains their central role (in that sense) in the theory of isospectral transforms.

To conclude, note that the two above elementary isospectral flows can coincide. As is well known (see for instance [16]), the reflectionless soliton potentials can be written in the form $U = 2\sqrt{2}\sum_{n=0}^{N} \sqrt{-E_n}\psi_n^2$ (where the sum is taken over all discrete states of the Schrödinger operator). Hence, for these potentials the MKT flow with weights $\tau_n^{(0)} = c\sqrt{-E_n}$ degenerates into the simplest shift flow; therefore, all the higher kth flows of the KdV hierarchy coincide with an MKT flow with proper weights $\tau_n^{(k)} = c\sqrt{-E_n} \cdot E_n^k$.

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References

[1] Darboux G 1882 *C. R. Acad. Sci., Paris* **94** 1456
[2] Poschel J and Trubowitz E 1987 *Inverse Spectral Theory* (New York: Academic)
[3] Crum M M 1955 *Q. J. Math.* **6** 121.
[4] Krein M G 1957 *Dokl. Akad. Nauk SSSR* **113** 970
[5] Nieto M M 1984 *Phys. Lett.* B **145** 208
[6] Luban M and Pursey D L 1986 *Phys. Rev.* D **33** 431
[7] Schrödinger E 1940 *Proc. R. Irish. Acad.* A **46** 9
Schrödinger E 1940 *Proc. R. Irish. Acad.* A **46** 183
Schrödinger E 1941 *Proc. R. Irish. Acad.* A **47** 53
[8] Infeld L and Hull E T 1951 *Rev. Mod. Phys.* **23** 21
[9] Mielnik B 1984 *J. Math. Phys.* **25** 3387
[10] Witten E 1981 *Nucl. Phys.* B **188** 513
[11] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** L57
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
Sukumar C V 1988 *J. Phys. A: Math. Gen.* **21** L455
[12] Veselov A P and Shabat A B 1993 *Funkz. Anal. Pril.* **27** 1
[13] Eleonsky V M and Korolev V G 1995 *J. Phys. A: Math. Gen.* **28** 4973
Eleonsky V M and Korolev V G 1996 *J. Phys. A: Math. Gen.* **29** L241
Eleonsky V M and Korolev V G 1996 *Sov. Phys.–JETP* **83** 1084
[14] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons: The Inverse Scattering Method* (New York: Consultants Bureau)
[15] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[16] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* vol I (Amsterdam: North-Holland)
[17] Poincaré H 1898 *J. Math. Pure. Appl.* **4** 137
[18] Santini P M 1996 *Phys. Lett.* A **212** 43
[19] Eleonsky V M and Korolev V G 1997 *Phys. Rev.* A **55** 2580
[20] McKean H P and Trubowitz E 1982 *Commun. Math. Phys.* **82** 471
[21] Levitan B M 1987 *Mat. Sbornik* **132** 73
[22] Eleonsky V M and Korolev V G *J. Math. Phys.* submitted
[23] Degasperis A and Sabatier P C 1987 *Inverse Problems* **3** 73
[24] Faddeev L D and Takhtajan L A 1985 *Hamiltonian Approach in Soliton Theory* (Berlin: Springer)